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k-gerbes, line bundles and anomalies

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Abstract: We use sets of trivial line bundles for the realization of gerbes. For 1-gerbes the structure arises naturally for the Weyl fermion vacuum bundle at a fixed time. The Schwinger term is an obstruction in the triviality of a 1-gerbe.

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Contents

1. Introduction

Gerbes were introduced by Giraud [1] in the early seventies. However, their importance in physics was not understood until about twenty years later when publications by authors such as Brylinski and Freed appeared, see for instance [2, 3]. Since then, the number of people working on gerbes has increased for each year. The importance of gerbes in string theory will probably lead to a continuation of this trend. One obstacle is the fact that the theory behind gerbes contains mathematics which is unfamiliar to most physicists. Even the basic definition of a gerbe, as a sheaf of groupoids with certain properties, is not easy to digest. This initiated the authors of [4, 5, 6] to use simple differential geometrical objects for the realization of gerbes. In these papers, and in the present, the following loose definition is used: a $(k-1)$-gerbe with band $\mathbb{C}^\times$ on a manifold B is a set of elements (which preferable have some geometrical meaning) with an equivalence relation such that the set of equivalence classes can be given a group structure to make it is isomorphic with the Čech cohomology $\check{H}^k(B, \underline{\mathbb{C}}^\times)$, $k \geq 2$. This definition is certainly not the same as the original one, but the 1-gerbes in [4, 5, 6] are actually also gerbes in Giraud's sense. Furthermore, the definition reduces to what we want for $k = 1$, i.e. that line bundles are classified by $\check{H}^1(B, \underline{\mathbb{C}}^\times)$.

In this paper, we will review and develop a construction that was attempted in [4], where certain sets of local (trivial) line bundles are interpreted as k-gerbes. Independently, this construction has also been carried out by Hitchin and Chatterjee for the case of 1-gerbes, [5]. Closely related to the construction are the bundle gerbes used by Murray [6]. Since it appears that some papers misinterpret the construction in [4, 5] we find it important to develope it in detail.

To motivate the importance of sets of local line bundles we will show how they appear naturally from the vacuum line bundle. A geometrical description of the vacuum and the Schwinger term in terms of gerbes has previously been done in [8] and in many other articles based on [6]. Although the construction in these papers is similar to our, it is more complicated and artificial for two reasons: firstly, bundle gerbes generalize to higher order gerbes in a complicated way, see for instance [7]. Secondly, to obtain an isomorphism with $\check{H}^2(B,\underline{\mathbb{C}}^\times)$ one has to use an equivalence of bundle gerbes in a sense similar to what is done in K-theory.

We will derive the Schwinger term from the curvature of the determinant line bundle for a manifold with boundary. The derivation is similar to the one made in [8]. We come to the conclusion that the Schwinger term, with the relevant cohomology, can be interpreted as an obstruction in a set of local line bundles on $\mathcal{A}/\mathcal{G}$ which is equipped with a certain structure. This is the analog of the geometrical description of the chiral anomaly in the space-time formalism given by Atiyah and Singer [9]. This shows that the description of gerbes as local line bundles is no abstract non sense and in fact appears in a natural way, at least for 1-gerbes.

We will start by proving that gerbes can be described by sets of local line bundles with a certain structure. This will be done in section 2. In section 3 we review the Čech-de Rham cohomology and in section 4 we pull back a part of it to an affine space so that new interesting cohomology groups are obtained. Using the mathematical results of sections 3 and 4 we show in section 5 that the geometrical structure which has been developed appears naturally in the context of anomalies and Schwinger terms.

2. Čech cohomology and sets of line bundles

In this section we will study the Čech cohomology $\check{H}^k(B,\underline{\mathbb{C}}^\times)$ of a manifold B. To get rid of the dependence of coverings we restrict ourselves to good (or Leray) coverings $\{U_\lambda\}_{\lambda\in\Lambda}$. We thus only consider manifolds B which admit such coverings. The existence of good coverings is obvious when B is finite dimensional, see for instance [10]. In general, however, we will allow B to be infinite dimensional. We also assume that B (and all other manifolds which we discuss) is smooth and Hausdorff. We will start by reviewing the fact that $\check{H}^1(B,\underline{\mathbb{C}}^\times)$ classifies line bundles.

Let $\mathcal{L}$ be a line bundle on B. Decompose it into trivial pieces $\mathcal{L}|_{U_\lambda}$, choose non-vanishing sections s_λ and define $g_{\lambda\mu}$ by

$$s_\lambda = s_\mu g_{\lambda\mu} \tag{2.1}$$

on $U_{\lambda\mu} = U_\lambda \cap U_\mu$. This equation is actually not well defined. The reason is that s_λ is a section of $\mathcal{L}|_{U_\lambda}$ while $s_\mu g_{\lambda\mu}$ is a section of $\mathcal{L}|_{U_\mu}$. We have an equality of sections belonging to different line bundles. The correct way to make sense of this is to introduce an isomorphism $\chi_{\lambda\mu} : (\mathcal{L}|_{U_\lambda})|_{U_\mu} \to (\mathcal{L}|_{U_\mu})|_{U_\lambda}$ and use it for identification. Then eq. (2.1) becomes:

$$\chi_{\lambda\mu}(s_\lambda) = s_\mu g_{\lambda\mu} \qquad \text{on } U_{\lambda\mu}\,. \tag{2.2}$$

This is however implicit in eq. (2.1) where $\chi_{\lambda\mu}$ has been chosen as the canonical isomorphism which is obtained from the fact that $\mathcal{L}|_{U_\lambda}$ and $\mathcal{L}|_{U_\mu}$ are restrictions of the same global bundle $\mathcal{L}$. From a line bundle we have thereby constructed a triple $(\mathcal{L}_\lambda, s_\lambda, \chi_{\lambda\mu})$, where $\chi_{\lambda\mu}$ in this case is a canonical isomorphism. The set $\{g_{\lambda\mu}\}$, defined by eq. (2.2), represents an element in $\check{H}^1(B, \underline{\mathbb{C}}^\times)$ that is independent of the choice of s_λ's, the covering and the choice of representative of the element $[\mathcal{L}]$ in the equivalence class of line bundles.

Conversely, let a Čech cocycle $\{g_{\lambda\mu}\}$ be given and choose line bundles $\mathcal{L}_\lambda \to U_\lambda$ with sections s_λ (the choices $\mathcal{L}_\lambda = U_\lambda \times \mathbb{C}$ and $s_\lambda = 1_\lambda$ are frequent in the literature). Then define $\chi_{\lambda\mu}$ by eq. (2.2). Again, we have constructed a triple $(\mathcal{L}_\lambda, s_\lambda, \chi_{\lambda\mu})$. Equation (2.2) can be regarded as an equivalence relation in the set $\{\mathcal{L}_\lambda\}$. By taking the quotent we obtain a smooth line bundle whose equivalence class is independent of the choices that were made.

The two maps that were described above are homomorphisms and each others inverses. We have thereby established an isomorphism between $\check{H}^1(B, \underline{\mathbb{C}}^\times)$ and the group of equivalence classes of line bundles. As a "middle step" in each of the homomorphisms, a triple appears. It is clear that if a certain equivalence relation and group structure is introduced on triples, we obtain another group which is isomorphic to $\check{H}^1(B, \underline{\mathbb{C}}^\times)$. That the class of $\{g_{\lambda\mu}\}$ is independent of $\{s_\lambda\}$ means that it is only equivalence classes of pairs $(\mathcal{L}_\lambda, \chi_{\lambda\mu})$ that are important from a cohomological viewpoint. We have thereby established:

$$\begin{aligned} \text{Čech cohomology} &\cong \{\text{Equivalence classes of pairs}\} \\ &\cong \{\text{Classes of line bundles}\}\,. \end{aligned} \tag{2.3}$$

We will soon show how parts of this can be generalized to $\check{H}^k(B, \underline{\mathbb{C}}^\times)$. First, we will however reinterpret $\chi_{\lambda\mu}$. This may appear strange at first, but in the generalization to arbitrary values of k the idea will become clear. We will equipp $\mathcal{L}_\lambda$ with a non-degenerate metric so that the dual $\mathcal{L}_\lambda^{-1}$ is well defined. Then s_λ defines a section s_λ^{-1} on $\mathcal{L}_\lambda^{-1}$ by $s_\lambda^{-1}(s_\lambda) = 1$. There is also a canonical isomorphism $\xi_\lambda : \mathcal{L}_\lambda \to \mathcal{L}_\lambda^{-1}$ given by $\xi_\lambda(s_\lambda) = s_\lambda^{-1}$ and $\xi_\lambda(s_\lambda r_\lambda) = s_\lambda^{-1} r_\lambda^{-1}$, where $r_\lambda : U_\lambda \to \mathbb{C}^\times$. Then $\chi_{\lambda\mu}$ defines a section $\sigma_{\lambda\mu}$ on $\mathcal{L}_\lambda \otimes \mathcal{L}_\mu^{-1}$ according to: $\sigma_{\lambda\mu} = s_\lambda \otimes \xi_\mu(\chi_{\lambda\mu}(s_\lambda))$. Clearly, $\sigma_{\lambda\mu}$ is independent of the choice of s_λ.

Let us now generalize the discussion to $\check{H}^k(B, \underline{\mathbb{C}}^\times)$. Let $\{\mathcal{L}_{\lambda_0..\lambda_{k-1}}\}$ and $\{s_{\lambda_0..\lambda_{k-1}}\}$ be (trivial) line bundles equipped with a non-degenerate hermitean metric and with corresponding non-vanishing sections over all $U_{\lambda_0\ldots\lambda_{k-1}} = U_{\lambda_0} \cap \cdots \cap U_{\lambda_{k-1}}$. As above, $s^{-1}_{\lambda_0\ldots\lambda_{k-1}}$ is given by $s^{-1}_{\lambda_0\ldots\lambda_{k-1}}(s_{\lambda_0\ldots\lambda_{k-1}}) = 1$ and $\{\xi_{\lambda_0\ldots\lambda_{k-1}}\}$ by $\xi_{\lambda_0\ldots\lambda_{k-1}}(s_{\lambda_0\ldots\lambda_{k-1}}) =$

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$s^{-1}_{\lambda_0\dots\lambda_{k-1}}$. Notice that in the construction that follows we will avoid discussing equality between the line bundles themselves, because all line bundles over a contractible set $U_{\lambda_0\dots\lambda_{k-1}}$ are isomorphic. We thus need an equivalence relation for sets of line bundles. This is provided by definition 2 below. We define $\{(\delta\mathcal{L})_{\lambda_0\dots\lambda_k}\}$ and $\{(\delta s)_{\lambda_0\dots\lambda_k}\}$ over $U_{\lambda_0\dots\lambda_k}$ as:

$$
\begin{aligned}
(\delta\mathcal{L})_{\lambda_0\dots\lambda_k} &:= \mathcal{L}_{\lambda_1\lambda_2\dots\lambda_k} \otimes \mathcal{L}_{\lambda_0\lambda_2\dots\lambda_k}{}^{-1} \otimes \dots \otimes \mathcal{L}_{\lambda_0\lambda_1\dots\lambda_{k-1}}{}^{(-1)^k}\,, \\
(\delta s)_{\lambda_0\dots\lambda_k} &:= s_{\lambda_1\lambda_2\dots\lambda_k} \otimes s_{\lambda_0\lambda_2\dots\lambda_k}{}^{-1} \otimes \dots \otimes s_{\lambda_0\lambda_1\dots\lambda_{k-1}}{}^{(-1)^k}\,,
\end{aligned}
\tag{2.4}
$$

where $a \otimes b \otimes c \otimes \dots := (((a \otimes b) \otimes c) \otimes \dots)$. It is implicit that the line bundles and sections on the right-hand side are restricted to $U_{\lambda_0\dots\lambda_k}$. Since $(\delta\mathcal{L}^{-1})_{\lambda_0\dots\lambda_k}$ can be identified with the dual of $(\delta\mathcal{L})_{\lambda_0\dots\lambda_k}$, it is clear how $\xi_{\lambda_0\dots\lambda_{k-1}}$ induces an isomorphism $(\delta\xi)_{\lambda_0\dots\lambda_k}$ from $(\delta\mathcal{L})_{\lambda_0\dots\lambda_k}$ to its dual. Notice that there exists a canonical section $(\delta^2 s)_{\lambda_0\dots\lambda_{k+1}}$ on $(\delta^2\mathcal{L})_{\lambda_0\dots\lambda_{k+1}}$. The section is independent of the choice of $\{s_{\lambda_0\dots\lambda_{k-1}}\}$. This makes it possible to canonically identify $(\delta^2\mathcal{L})_{\lambda_0\dots\lambda_{k+1}}$ with the product bundle $U_{\lambda_0\dots\lambda_{k+1}} \times \mathbb{C}$ (this bundle has a canonical section taking the value $1 \in \mathbb{C}$). We will therefore think of line bundles of the type $(\delta^2\mathcal{L})_{\lambda_0\dots\lambda_{k+1}}$ as the identity. We will then consider the cohomology generated when δ acts on sets $\{(\mathcal{L}'_{\lambda_0\dots\lambda_k}, s'_{\lambda_0\dots\lambda_k})\}$ according to (2.4). Thus, the cocycles should be such that δ acting on them will give the identity $\{(\delta^2\mathcal{L})_{\lambda_0\dots\lambda_{k+1}}, (\delta^2 s)_{\lambda_0\dots\lambda_{k+1}}\}$, for some set $\{(\mathcal{L}_{\lambda_0\dots\lambda_{k-1}}, s_{\lambda_0\dots\lambda_{k-1}})\}$. It follows that $\mathcal{L}'_{\lambda_0\dots\lambda_k} := (\delta\mathcal{L})_{\lambda_0\dots\lambda_k}$ and $(\delta s')_{\lambda_0\dots\lambda_{k+1}} = (\delta^2 s)_{\lambda_0\dots\lambda_{k+1}}$. We have thereby motivated the following definitions:

Definition 1 A set $\{(\mathcal{L}_{\lambda_0\dots\lambda_{k-1}}, \sigma_{\lambda_0\dots\lambda_k})\}$ of pairs consists of line bundles $\mathcal{L}_{\lambda_0\dots\lambda_{k-1}}$ (equipped with a non-degenerate hermitean metric) on each $U_{\lambda_0\dots\lambda_{k-1}}$ and non-vanishing sections $\sigma_{\lambda_0\dots\lambda_k}$ on each $(\delta\mathcal{L})_{\lambda_0\dots\lambda_k}$ such that $(\delta\sigma)_{\lambda_0\dots\lambda_{k+1}} = (\delta^2 s)_{\lambda_0\dots\lambda_{k+1}}$.

Definition 2 $\{(\mathcal{L}_{\lambda_0\dots\lambda_{k-1}}, \sigma_{\lambda_0\dots\lambda_k})\}$ and $\{(\mathcal{L}'_{\lambda_0\dots\lambda_{k-1}}, \sigma'_{\lambda_0\dots\lambda_k})\}$ are said to be equivalent if there exists a diffeomorphism between them, i.e. diffeomorphisms $\varphi_{\lambda_0\dots\lambda_{k-1}} : \mathcal{L}_{\lambda_0\dots\lambda_{k-1}} \to \mathcal{L}'_{\lambda_0\dots\lambda_{k-1}}$ such that $(\delta\varphi)_{\lambda_0\dots\lambda_k}(\sigma_{\lambda_0\dots\lambda_k}) = \sigma'_{\lambda_0\dots\lambda_k}$.

$(\delta\varphi)_{\lambda_0\dots\lambda_k}$ is defined in the obvoius way with $\varphi^{-1}_{\lambda_0\dots\lambda_{k-1}}$ defined as $\xi'_{\lambda_0\dots\lambda_{k-1}} \circ \varphi_{\lambda_0\dots\lambda_{k-1}} \circ \xi^{-1}_{\lambda_0\dots\lambda_{k-1}}$. Two sets of pairs can be multiplied to give $\{(\mathcal{L}_{\lambda_0\dots\lambda_{k-1}} \otimes \mathcal{L}'_{\lambda_0\dots\lambda_{k-1}}, (\sigma \otimes \sigma')_{\lambda_0\dots\lambda_k})\}$, where $(\sigma \otimes \sigma')_{\lambda_0\dots\lambda_k}$ is the image of $\sigma_{\lambda_0\dots\lambda_k} \otimes \sigma'_{\lambda_0\dots\lambda_k}$ under the canonical isomorphism from $(\delta\mathcal{L})_{\lambda_0\dots\lambda_k} \otimes (\delta\mathcal{L}')_{\lambda_0\dots\lambda_k}$ to $(\delta(\mathcal{L} \otimes \mathcal{L}'))_{\lambda_0\dots\lambda_k}$. This defines an abelian group $H^k(\mathcal{L}, \sigma)$ whose elements are the equivalence classes of (sets of) pairs. The identity element consists of pairs $(\mathcal{L}_{\lambda_0\dots\lambda_{k-1}}, \sigma_{\lambda_0\dots\lambda_k})$ such that there exist sections $s_{\lambda_0\dots\lambda_{k-1}}$ with $(\delta s)_{\lambda_0\dots\lambda_k} = \sigma_{\lambda_0\dots\lambda_k}$.

Proposition 1 $H^k(\mathcal{L}, \sigma) \cong \check{H}^k(B, \underline{\mathbb{C}}^\times)$.

Proof. This follows directly from the equation

$$
(\delta s)_{\lambda_0\dots\lambda_k} = \sigma_{\lambda_0\dots\lambda_k} g_{\lambda_0\dots\lambda_k}\,,
\tag{2.5}
$$

where $s_{\lambda_0\dots\lambda_{k-1}}$ is a non-vanishing section of $\mathcal{L}_{\lambda_0\dots\lambda_{k-1}}$. □

This result implies that $H^k(\mathcal{L},\sigma)$ is independent of the covering, as indicated by the notation. We have thus verified the left isomorphism in (2.3) for all $k \geq 1$. Unfortunately, we have not been able to establish an equivalence of the isomorphism to the right. However, this was not to be expected since we started off with line bundles which obviously are the wrong geometrical objects for this to hold. We just have to accept that we can only come half-way in (2.3) and instead try to make the best of it.

Let us consider the case $k = 0$. For this case a pair $\{(\mathcal{L},\sigma_\lambda)\}$ consists by definition of a trivial line bundle $\mathcal{L}$ over B and non-vanishing sections σ_λ of $\mathcal{L}_\lambda = (\delta\mathcal{L})_\lambda = \mathcal{L}|_{U_\lambda}$. A pair is a cocycle if $(\delta\sigma)_{\lambda\mu} = (\delta^2 s)_{\lambda\mu}$. Notice that $(\delta^2 s)_{\lambda\mu}$ is the canonical section of $\mathcal{L}_\mu \otimes \mathcal{L}_\lambda^{-1}$ given by $s|_{U_\mu} \otimes (s|_{U_\lambda})^{-1}$. Thus, the cocycle relation means that the σ's glue together to define a global section. The corresponding Čech cocycle, given by $s|_{U_\lambda} = \sigma_\lambda g_\lambda$, then also glue together to define a smooth $\mathbb{C}^\times$-valued function on B. Thus, we see that elements in $\check{H}^0(B,\underline{\mathbb{C}}^\times)$, which are $\mathbb{C}^\times$-valued functions on B, can be identified with the "quotient" of two non-vanishing sections of a trivial bundle on B. Notice that the case $k = 0$ is special in our setting: it is necessary to introduce a fixed choice of section s on $\mathcal{L}$ (because g depends on s).

We have already considered $k = 1$, so lets turn to the more complicated case when $k \geq 2$. One way to get information about how (sets of) pairs look in these cases is to study the case when B is contractible. Then everything becomes trivial so a degree k cocycle can be written as the coboundary operator acting on a $k-1$ cochain, which often is simpler to understand. Consider for example the case $k = 1$. In this case triviality means that there exists a representative $\{(\mathcal{L}_\lambda,\sigma_{\lambda\mu})\}$ of a cohomology class of pairs which is of the form $\delta\{(\mathcal{L}',\sigma'_\lambda)\}$. That $\mathcal{L}_\lambda \equiv (\delta\mathcal{L}')_\lambda$ means that the $\mathcal{L}_\lambda$'s are restrictions of a global trivial bundle $\mathcal{L}'$. That a line bundle over a contractible space is a trivial line bundle we already knew, but here the setting of pairs gave us the result. When $k = 2$ triviality means that a representative $\{(\mathcal{L}_{\lambda\mu},\sigma_{\lambda\mu\nu})\}$ of a cohomology class of pairs is of the form $\delta\{(\mathcal{L}'_\lambda,\sigma'_{\lambda\mu})\}$. Thus, in this case we get: $\mathcal{L}_{\lambda\mu} \equiv \mathcal{L}'_\mu \otimes {\mathcal{L}'_\lambda}^{-1}$ and $\sigma_{\lambda\mu\nu} \equiv \sigma'_{\mu\nu} \otimes {\sigma'_{\lambda\nu}}^{-1} \otimes \sigma'_{\lambda\mu}$. Observe that $\{(\mathcal{L}'_\lambda,\sigma'_{\lambda\mu})\}$ does not describe a global line bundle since it is not a cocycle. The "transition functions" do not obey the cocycle relation and the local lines therefore do not glue together.

The pairs $(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}},\sigma_{\lambda_0\ldots\lambda_k})$ are actually independent of the first argument $\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}$. The line bundles are only needed to define the $\sigma_{\lambda_0\ldots\lambda_k}$'s and can thus be chosen to be product bundles. Thus, the Čech cohomology can be interpreted as certain objects $(\{\sigma_{\lambda_0\ldots\lambda_k}\})$ which determine how local product bundles "glue together". However, a product bundle is not just a line bundle, but a line bundle with a choice of section. To avoid introducing additional structure, we will therefore stick to pairs.

It is possible to drop the condition that $\{U_\lambda\}$ should be a good covering as long as it is assumed that the line bundles $\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}$ and $(\delta\mathcal{L})_{\lambda_0\ldots\lambda_k}$ are trivial. In this case we obtain a cohomology group $H^k(\{U_\lambda\},\mathcal{L},\sigma)$ which is isomorphic to the covering-dependent Čech cohomology $\check{H}^k(\{U_\lambda\},\underline{\mathbb{C}}^\times)$. Let us also point out that the constructions in this section go through when the line bundles are replaced by principal bundles with abelian fibres, i.e. $\underline{\mathbb{C}}^\times$ can be replaced with any other abelian group. Notice however that $\delta\mathcal{L}$ has a different fibre than $\mathcal{L}$ for that case.

3. The Čech-de Rham complex

The only assumptions that will be made in this section is that B admits a partition of unity subordinate to a good covering if it is finite dimensional. If B is infinite dimensional, some additional assumptions will be needed, see [11]. These assumptions will however not be discussed here.

Let us for a moment ignore the torsion part in $\check{H}^k(B,\underline{\mathbb{C}}^\times)$. Then the Čech-de Rham complex gives an injective map from $\check{H}^k(B,\underline{\mathbb{C}}^\times)$ to $H^{k+1}_{dR}(B,\underline{\mathbb{C}})$. Thus, $\check{H}^k(B,\underline{\mathbb{C}}^\times)$ is isomorphic to a subgroup $H^{k+1}_{dR,0}(B,\underline{\mathbb{C}})$ of the de Rham cohomology. The subgroup is characterized by

Proposition 2 $[F]\in H^{k+1}_{dR,0}(B,\underline{\mathbb{C}})\Leftrightarrow F$ *satisfies the integrality condition.*

The proof of this statement is given in appendix A. Recall that the Čech-de Rham complex not only helps to prove $\check{H}^k(B,\underline{\mathbb{C}}^\times)\cong H^{k+1}_{dR,0}(B,\underline{\mathbb{C}})$, but it also shows that the "path" between the two cohomologies can be made step-wise. When "walking" through the Čech-de Rham complex the degree of the Čech cocycles are decreased at the expense of an equal increase of the form degree (or vice versa). There will appear intermediate groups which are built up by elements which are mixtures of closed de Rham forms and Čech cocycles. This can be of advantage for us when trying to understand $\check{H}^k(B,\underline{\mathbb{C}}^\times)$. For instance, closed global 3-forms can be turned into a set of closed local 2-forms. This gives a geometrical realization of the free part of $\check{H}^2(B,\underline{\mathbb{C}}^\times)$ since closed local 2-forms can be considered as curvatures of local line bundles.

In section 2 we obtained a realization of the elements in $\check{H}^k(B,\underline{\mathbb{C}}^\times)$ in terms of sets of pairs of line bundles and sections. We will now show that a similar construction can be made for $\check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})_{(k,1)}$ and $\check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})_{(k-1,2)}$ (which are both isomorphic to $\check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})$). The index (p,q) refers to the part of the Čech-de Rham cohomology that is built up by q-forms on each $U_{\lambda_0\ldots\lambda_p}$. We will consider sets $\{(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}},\nabla_{\lambda_0\ldots\lambda_{k-1}})\}$ of pairs of line bundles with connections. As usual, the line bundles $\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}$ are assumed to be equipped with a non-degenerate metric.

Definition 3 $\{(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}},\nabla_{\lambda_0\ldots\lambda_{k-1}})\}$ and $\{(\mathcal{L}'_{\lambda_0\ldots\lambda_{k-1}},\nabla'_{\lambda_0\ldots\lambda_{k-1}})\}$ are said to be equivalent if there exists a diffeomorphism between them, i.e. diffeomorphisms $\varphi_{\lambda_0\ldots\lambda_{k-1}}:\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}\to\mathcal{L}'_{\lambda_0\ldots\lambda_{k-1}}$ such that

$$\varphi_{\lambda_0\ldots\lambda_{k-1}}(\nabla_{\lambda_0\ldots\lambda_{k-1},v}s_{\lambda_0\ldots\lambda_{k-1}})=\nabla'_{\lambda_0\ldots\lambda_{k-1},v}\varphi_{\lambda_0\ldots\lambda_{k-1}}(s_{\lambda_0\ldots\lambda_{k-1}}),$$

for any tangent field v on B and any section $s_{\lambda_0\ldots\lambda_{k-1}}$ on $\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}$.

Multiplication of pairs gives by definition $\{(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}\otimes\mathcal{L}'_{\lambda_0\ldots\lambda_{k-1}},\nabla_{\lambda_0\ldots\lambda_{k-1}}+\nabla'_{\lambda_0\ldots\lambda_{k-1}})\}$, where

$$\nabla_{\lambda_0\ldots\lambda_{k-1}}+\nabla'_{\lambda_0\ldots\lambda_{k-1}}(s_{\lambda_0\ldots\lambda_{k-1}}\otimes s'_{\lambda_0\ldots\lambda_{k-1}}) = (\nabla_{\lambda_0\ldots\lambda_{k-1}}s_{\lambda_0\ldots\lambda_{k-1}})\otimes s'_{\lambda_0\ldots\lambda_{k-1}} + \\ + s_{\lambda_0\ldots\lambda_{k-1}}\otimes(\nabla'_{\lambda_0\ldots\lambda_{k-1}}s'_{\lambda_0\ldots\lambda_{k-1}})\,.$$

This defines an abelian group $C^{k+1}(\{U_\lambda\},\mathcal{L},\nabla)$ whose elements are the equivalence classes of (sets of) pairs. The identity element clearly consists of sets of pairs $\{(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}},\nabla_{\lambda_0\ldots\lambda_{k-1}})\}$ such that there exist non-vanishing horizontal (flat) sections, i.e. sections satisfying $\nabla_{\lambda_0\ldots\lambda_{k-1}}s_{\lambda_0\ldots\lambda_{k-1}}=0$. Every element in $C^{k+1}(\{U_\lambda\},\mathcal{L},\nabla)$ defines a Čech-de Rham cochain $\{(\delta\omega)_{\lambda_0\ldots\lambda_k}\}$ up to a coboundary by

$$\nabla_{\lambda_0\ldots\lambda_{k-1}}s_{\lambda_0\ldots\lambda_{k-1}} = s_{\lambda_0\ldots\lambda_{k-1}}\omega_{\lambda_0\ldots\lambda_{k-1}}\,, \tag{3.1}$$

where $s_{\lambda_0\ldots\lambda_{k-1}}$ is a non-vanishing section of $\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}}$. The map

$$\left(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}},\nabla_{\lambda_0\ldots\lambda_{k-1}}\right)\longmapsto((\delta\mathcal{L})_{\lambda_0\ldots\lambda_k},(\delta\nabla)_{\lambda_0\ldots\lambda_k})$$

induces a homomorphism $\delta : C^{k+1}(\{U_\lambda\},\mathcal{L},\nabla)\to C^{k+2}(\{U_\lambda\},\mathcal{L},\nabla)$. The operator $(\delta\nabla)_{\lambda_0\ldots\lambda_k}$ is defined in the obvious way. Notice that $(\delta^2 s)_{\lambda_0\ldots\lambda_{k+1}}$ is horizontal with respect to $(\delta^2\nabla)_{\lambda_0\ldots\lambda_{k+1}}$. Since δ^2 gives the identity element we are lead to:

Definition 4 Let $H^{k+1}(\mathcal{L},\nabla)$ be the cohomology defined by $C^{k+1}(\{U_\lambda\},\mathcal{L},\nabla)$ and δ. $H^{k+1}_0(\mathcal{L},\nabla)$ denotes the subgroup built up by pairs with horizontal sections $\sigma_{\lambda_0\ldots\lambda_k}$ of $(\delta\mathcal{L})_{\lambda_0\ldots\lambda_k}$ with respect to $(\delta\nabla)_{\lambda_0\ldots\lambda_k}$ such that $(\delta\sigma)_{\lambda_0\ldots\lambda_{k+1}}$ equals the canonical section $(\delta^2 s)_{\lambda_0\ldots\lambda_{k+1}}$.

Equation (3.1) gives $H^{k+1}(\mathcal{L},\nabla)\cong\check{H}^{k+1}_{dR}(B,\underline{\mathbb{C}})_{(k,1)}$ and restricts to $H^{k+1}_0(\mathcal{L},\nabla)\cong\check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})_{(k,1)}$. The latter isomorphism can be proven by using the fact that $H^{k+1}_0(\mathcal{L},\nabla)$ is isomorphic to the free part $H^k_0(\mathcal{L},\sigma)$ of $H^k(\mathcal{L},\sigma)$. To show this it is helpful to use the fact that $H^k_0(\mathcal{L},\sigma)\cong\check{H}^k_0(B,\underline{\mathbb{C}}^\times)$, where the latter group is $\check{H}^k(B,\underline{\mathbb{C}}^\times)$ modulo constant cocycles. Every $[\{(\mathcal{L}_{\lambda_0\ldots\lambda_{k-1}},\nabla_{\lambda_0\ldots\lambda_{k-1}})\}]\in H^{k+1}(\mathcal{L},\nabla)$ defines an element $[\{F_{\lambda_0\ldots\lambda_{k-1}}\}]\in\check{H}^{k+1}_{dR}(B,\underline{\mathbb{C}})_{(k-1,2)}$ by:

$$(\nabla^2)_{\lambda_0\ldots\lambda_{k-1}}s_{\lambda_0\ldots\lambda_{k-1}} = s_{\lambda_0\ldots\lambda_{k-1}}F_{\lambda_0\ldots\lambda_{k-1}}\,, \tag{3.2}$$

where $(\nabla^2)_{\lambda_0\ldots\lambda_{k-1}}$ is the curvature of $\nabla_{\lambda_0\ldots\lambda_{k-1}}$. This induces an isomorphism $H^{k+1}(\mathcal{L},\nabla)\cong\check{H}^{k+1}_{dR}(B,\underline{\mathbb{C}})_{(k-1,2)}$, which in turn restricts to $H^{k+1}_0(\mathcal{L},\nabla)\cong\check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})_{(k-1,2)}$. We summarize the isomorphisms above in the following diagram:

Lemma 1

$$\begin{array}{ccccc} H^k_0(\mathcal{L},\sigma) & \cong & H^{k+1}_0(\mathcal{L},\nabla) & = & H^{k+1}_0(\mathcal{L},\nabla) \\ \cong & & \cong & & \cong \\ \check{H}^k_0(B,\underline{\mathbb{C}}^\times) & \cong & \check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})_{(k,1)} & \cong & \check{H}^{k+1}_{dR,0}(B,\underline{\mathbb{C}})_{(k-1,2)}\,. \end{array}$$

4. The pull-back to an affine space

Let $\mathcal{A}$ be an affine space and $\pi : \mathcal{A} \to B$ a fibre bundle. We will see how some parts of the Čech-de Rham complex can be pulled-back from B to $\mathcal{A}$. The reason why not all elements in the complex can be pulled-back is that $\{\pi^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is not necessary a good covering. The advantage of the pull-back is that the form degrees will be of one order less, see below. Thus, the case $k = 2$ can then be treated in almost the same way as the simple case when $k = 1$. Some difficulties will remain however. Unfortunately, the case $k = 3$ will not become simpler than the case $k = 2$ was from the beginning. Our focus will therefore be on the cases $k = 1$ and $k = 2$. An interesting feature is that the integrality condition is not invariant under such a pull-back (or transgression). This stems from the existence of lower homotopy groups in the base manifold B.

The idea of pulling-back, or transgressing gerbes on the base of a fibre bundle is not new. It was used in [12] for 1- and 2-gerbes on the base manifold, see also [13, 14, 15]. What is new in our approach is to use an affine space as the total space. In the next section we will see that this is the natural approach in the context of anomalies (and Schwinger terms).

Often, we will put a hat on symbols referring to line bundles, forms, etc. when they are defined with respect to $\mathcal{A}$. Exceptions of this rule are the forms $\hat{a}$ and $\hat{\eta}$ that appear in the next section.

Definition 5 Denote by $\Omega^k(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ the quotient group of $\Omega^k(\mathcal{A}, \underline{\mathbb{C}})$ by forms of type $\pi^*\omega$. Then $H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ is the cohomology defined by $\Omega^k(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ and the coboundary operator (induced by) $\hat{d}$. Let $H^k_{dR,0}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ be the subgroup of representatives such that the integral over any closed k-manifold in a fibre gives an element in $2\pi i\mathbf{Z}$.

Lemma 2 $H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}}) \cong H^{k+1}_{dR}(B, \underline{\mathbb{C}})$

Proof. For $[F] \in H^{k+1}_{dR}(B, \underline{\mathbb{C}})$ the Poincaré lemma gives the existence of an $[\hat{\omega}] \in H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ so that $\hat{d}\hat{\omega} = \pi^*F$. The ambiguity in the choice of $\hat{\omega}$ and triviality of F both give coboundaries so a map $H^{k+1}_{dR}(B, \underline{\mathbb{C}}) \to H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ has been obtained. Since it is linear, surjective and invertible it is also an isomorphism. □

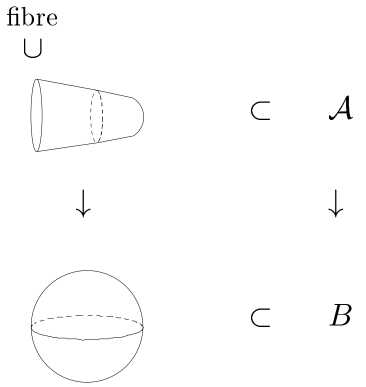
Lemma 3 *There exist a monomorphism* $H^{k+1}_{dR,0}(B, \underline{\mathbb{C}}) \to H^k_{dR,0}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$

Proof. The monomorphism is the isomorphism in lemma 2, but with a different domain and range. The only thing that needs to be checked is that $\hat{\omega}$ satisfies the integrality condition if F does. Let therefore S be an arbitrary closed k-manifold contained in a fibre. Since $\mathcal{A}$ is affine there exist a manifold V with $\partial V = S$. Then:

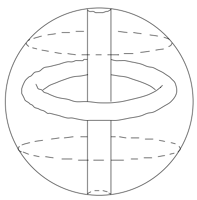
$$\int_S \hat{\omega} = \int_V \pi^* F = \int_{\pi(V)} F$$

which is in $2\pi i\mathbf{Z}$ since $\pi(V)$ is a manifold without boundary. □

The proof above uses ideas from a proof of the exact homotopy sequence of a fibre bundle, see [16, chapter 4]. For instance, the following fact is important: the group of homotopy classes of maps from the k-sphere: $S^k \to B$ is in bijective correspondence with the group of homotopy classes of maps $D^{k+1} \to \mathcal{A}$ such that ∂D^{k+1} is mapped into a fibre, see figure below:



The integral of F over a manifold $\pi(V)$ can only be non zero if there does not exist any manifold which has $\pi(V)$ as its boundary (since F is closed). The reason why we only obtained a monomorphism in lemma 3 is that there are "more" $(k+1)$-manifolds on B of this type than k-manifolds in the fibre of the same type. This happens even for $k = 1$: every 2-sphere on B lifts to a bowl on $\mathcal{A}$ which restricts to a 1-sphere in the fibre, and vice versa. This proves that $\pi_1(\text{fibre}) = \pi_2(B)$. However, there are more types of manifolds on B such that the integral of F will be non zero. This follows from the existence of lower homotopy groups. Consider for example the manifold B pictured below. It looks like an apple where a worm has made a hole from the top to the bottom and then (somehow) a torus around the original hole. By taking $\pi(V)$ as the surface of the torus we see that there is no closed manifold in the fibre that corresponds to it. To summarize: there is an injective map from the set of homotopy classes of closed k-manifolds in a fibre to the set of homotopy classes of closed $(k+1)$-manifolds on the base.



The discussion above is not completely honest. When considering homotopy groups one also has to take into account the dependence on the (base-) points for which the homotopy group is defined, [16]. However, when the space is connected there is no such dependence. Clearly, B is path-connected: take two points in B, lift them to the affine space $\mathcal{A}$ and then project down a path joining the two lifted points. For the fibre there is no corresponding trick, so here we will get a dependence on the choice of base point for the homotopy groups.

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Definition 6 Denote by $\tilde{\Omega}^k(\{U_\lambda\}, \mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ the abelian group of elements $\{\tilde{\omega}_\lambda\}$ consisting of k-forms over $\pi^{-1}(U_\lambda)$ of type:

$$\tilde{\omega}_\lambda = (\hat{\delta}\hat{\omega})_\lambda - \pi^*\omega_\lambda \,. \tag{4.1}$$

The cohomology $\tilde{H}^k(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ is then defined by $\tilde{\Omega}^k(\{U_\lambda\}, \mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ and the coboundary operator $\hat{d}$. Define $\tilde{H}^k_0(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ to be the subgroup of representatives $\tilde{\omega}_\lambda$ that satisfies the integrality condition with respect to k-manifolds in $\pi^{-1}(U_\lambda)$.

Lemma 4 *$\tilde{H}^k(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ is isomorphic to each of the grops $\check{H}^{k+1}_{dR}(B, \underline{\mathbb{C}})_{(1,k)}$, $\check{H}^{k+1}_{dR}$ $(B, \underline{\mathbb{C}})_{(0,k+1)}$ and $H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$.*

Proof. Applying $\hat{d}$ to a representative $\{\tilde{\omega}_\lambda\}$ of $\tilde{H}^k(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ gives:

$$(\hat{\delta}\hat{d}\hat{\omega})_\lambda = \pi^* d\omega_\lambda \,,$$

which implies that $\pi^*(\delta d\omega)_{\lambda\mu} = 0$ so $d\omega_\lambda = (\delta F)_\lambda$ for some F. Thus $[\{d\omega_\lambda\}] \in \check{H}^{k+1}_{dR}(B, \underline{\mathbb{C}})_{(0,k+1)}$ and $[\{(\delta\omega)_{\lambda\mu}\}] \in \check{H}^{k+1}_{dR}(B, \underline{\mathbb{C}})_{(1,k)}$. The equation above then becomes $(\hat{\delta}\hat{d}\hat{\omega})_\lambda = (\hat{\delta}\pi^* F)_\lambda$, or equivalently $\hat{d}\hat{\omega} = \pi^* F$, which implies that $[\hat{\omega}] \in H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$. It is straightforward to check that this gives the claimed isomorphisms. We will be settled by showing that the isomorphisms are independent of the decomposition in (4.1). Let $\tilde{\omega}_\lambda = (\hat{\delta}\hat{\omega}')_\lambda - \pi^*\omega'_\lambda$ be a different decomposition and set $\triangle\hat{\omega} = \hat{\omega}' - \hat{\omega}$ and $\triangle\omega_\lambda = \omega'_\lambda - \omega_\lambda$. Then

$$\pi^* \sum_\lambda \rho_\lambda \triangle\omega_\lambda = \sum_\lambda \pi^*\rho_\lambda \pi^* \triangle\omega_\lambda = \sum_\lambda \pi^*\rho_\lambda (\hat{\delta}\triangle\hat{\omega})_\lambda = \triangle\hat{\omega} \sum_\lambda \pi^*\rho_\lambda = \triangle\hat{\omega}$$

which shows that $\hat{\omega}'$ and $\hat{\omega}$ represents the same element in $H^k_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$. Further,

$$\pi^*\triangle\omega_\lambda = \left(\hat{\delta}\triangle\hat{\omega})_\lambda = \pi^*(\hat{\delta}\sum_\mu \rho_\mu \triangle\omega_\mu \right)_\lambda$$

which shows that the images in $\check{H}^{k+1}_{dR}(B, \underline{\mathbb{C}})_{(1,k)}$ respective $\check{H}^{k+1}_{dR}(B, \underline{\mathbb{C}})_{(0,k+1)}$ are independent of the decomposition. □

Although $\{\pi^{-1}(U_\lambda)\}$ is not a good covering we obtain as a corollary that $\tilde{H}^k(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ is independent of the covering (as long as it is good on B).

Lemma 5 $\tilde{H}^k_0(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}}) \cong H^k_{dR,0}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$.

Proof. Let $S \subset \pi^{-1}(U_\lambda)$ be a closed k-manifold. Then $\pi(S)$ is a closed manifold in the contractible set U_λ and therefore there exist a manifold $\pi(V)$ with $\partial\pi(V) = \pi(S)$. Since $\pi(V)$ can be obtained from a manifold V on $\pi^{-1}(U_\lambda)$, the notation is justified.

V can be chosen in such a way that $\partial V = S \cup S'$, where S' is such that $\pi(S')$ is a point. With notations from the proof of lemma 4 we get:

$$\begin{aligned}\int_S \tilde{\omega}_\lambda &= \int_S \Big((\hat{\delta}\hat{\omega})_\lambda - \pi^*\omega_\lambda \Big) = \int_V \pi^* F - \int_{S'} \hat{\omega} - \int_{\pi(S)} \omega_\lambda \\ &= \int_{\pi(V)} F - \int_{S'} \hat{\omega} - \int_{\pi(V)} F = -\int_{S'} \hat{\omega}\,.\end{aligned}$$

The converse part of the proof is obvious. □

Putting together the Čech-de Rham complex with lemmas 1–5, the statement below follows:

Theorem 1

$$\begin{array}{ccccccc} & H_0^k(\mathcal{L},\sigma) & \cong & H_0^{k+1}(\mathcal{L},\nabla) & = & H_0^{k+1}(\mathcal{L},\nabla) & \\ & \cong & & \cong & & \cong & \\ & \check{H}_0^k(B,\underline{\mathbb{C}}^\times) & \cong & \check{H}_{dR,0}^{k+1}(B,\underline{\mathbb{C}})_{(k,1)} & \cong & \check{H}_{dR,0}^{k+1}(B,\underline{\mathbb{C}})_{(k-1,2)} & \cong \\ \cong & \check{H}_{dR,0}^{k+1}(B,\underline{\mathbb{C}})_{(k-2,3)} & \cong & \cdots & \cong & \check{H}_{dR,0}^{k+1}(B,\underline{\mathbb{C}})_{(2,k-1)} & \cong \\ & \tilde{H}_0^k(\mathcal{A}\xrightarrow{\pi}B,\underline{\mathbb{C}}) & = & \tilde{H}_0^k(\mathcal{A}\xrightarrow{\pi}B,\underline{\mathbb{C}}) & \cong & H_{dR,0}^k(\mathcal{A}\xrightarrow{\pi}B,\underline{\mathbb{C}}) & \\ & \uparrow \text{monic} & & \uparrow \text{monic} & & \uparrow \text{monic} & \\ \cong & \check{H}_{dR,0}^{k+1}(B,\underline{\mathbb{C}})_{(1,k)} & \cong & \check{H}_{dR,0}^{k+1}(B,\underline{\mathbb{C}})_{(0,k+1)} & \cong & H_{dR,0}^{k+1}(B,\underline{\mathbb{C}})\,. & \end{array}$$

This becomes particular interesting when $k = 1$ or 2. The reason is that there exist rather simple geometrical realizations of $H^1_{dR,0}(\mathcal{A}\xrightarrow{\pi}B,\underline{\mathbb{C}})$ and $\tilde{H}^2_0(\mathcal{A}\xrightarrow{\pi}B,\underline{\mathbb{C}})$. In the following definition we use that $C^k(\{\pi^{-1}(U_\lambda)\},\hat{\mathcal{L}},\hat{\nabla})$, $k \geq 1$, can be defined also when the line bundles $\mathcal{L}_{\lambda_0\ldots\lambda_{k-2}}$ are non trivial.

Definition 7 Let $\tilde{C}^k(\{\pi^{-1}(U_\lambda)\},\hat{\mathcal{L}},\hat{\nabla})$ be the quotient of $C^k(\{\pi^{-1}(U_\lambda)\},\hat{\mathcal{L}},\hat{\nabla})$ by the group generated by elements which have a representative consisting of pairs of type $\{(\pi^*\mathcal{L}_{\lambda_0\ldots\lambda_{k-2}},\pi^*\nabla_{\lambda_0\ldots\lambda_{k-2}})\}$. The cohomology $\tilde{H}^k(\hat{\mathcal{L}},\hat{\nabla})$ is then defined by $\tilde{C}^k(\{\pi^{-1}(U_\lambda)\},\hat{\mathcal{L}},\hat{\nabla})$ and the coboundary operator $\hat{\delta}$ induced by the corresponding operator acting on $C^k(\{\pi^{-1}(U_\lambda)\},\hat{\mathcal{L}},\hat{\nabla})$.

Let us first focus on the case $k = 1$.

Lemma 6 $\tilde{H}^1(\hat{\mathcal{L}},\hat{\nabla}) \cong H^1_{dR}(\mathcal{A}\xrightarrow{\pi}B,\underline{\mathbb{C}})$

Proof. When $k = 1$, the representatives of $\tilde{C}^k(\{\pi^{-1}(U_\lambda)\},\hat{\mathcal{L}},\hat{\nabla})$ are line bundles $\hat{\mathcal{L}}$ with connections $\hat{\nabla}$, globally defined on $\mathcal{A}$. It is trivial if $(\hat{\mathcal{L}},\hat{\nabla})$ is equivalent with $(\pi^*\mathcal{L},\pi^*\nabla)$ for some trivial bundle $\mathcal{L}$ on B with connection ∇. Since $\mathcal{A}$ is affine, there exist a non-vanishing section $\hat{s}$ of $\hat{\mathcal{L}}$. Define $\hat{\omega}$ by:

$$\hat{\nabla}\hat{s} = \hat{s}\hat{\omega}\,. \tag{4.2}$$

The cocycle condition implies that $\hat{d}\hat{\omega} = \pi^* F$ for some $[F] \in H^2_{dR}(B, \underline{\mathbb{C}})$. It is straightforward to check that this defines an isomorphism from $\tilde{H}^1(\hat{\mathcal{L}}, \hat{\nabla})$ to $H^1_{dR}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$. □

Although the following statement is a consequence of theorem 1 and lemma 6, we will give a direct proof.

Lemma 7 $\tilde{H}^1(\hat{\mathcal{L}}, \hat{\nabla}) \cong H^2(\mathcal{L}, \nabla)$

Proof. Let $[(\hat{\mathcal{L}}, \hat{\nabla})] \in \tilde{H}^1(\hat{\mathcal{L}}, \hat{\nabla})$ be given. The cocycle condition states that $((\delta\hat{\mathcal{L}})_\lambda, (\delta\hat{\nabla})_\lambda)$ is equivalent to some $(\pi^*\mathcal{L}_\lambda, \pi^*\nabla_\lambda)$. This gives a well-defined map to $[\{(\mathcal{L}_\lambda, \nabla_\lambda)\}] \in H^2(\mathcal{L}, \nabla)$. Indeed, since $((\delta^2\hat{\mathcal{L}})_{\lambda\mu}, (\delta^2\hat{\nabla})_{\lambda\mu})$ allows a non-vanishing horizontal section, the same is true for $((\delta\mathcal{L})_{\lambda\mu}, (\delta\nabla)_{\lambda\mu})$. Further, $[\{(\mathcal{L}_\lambda, \nabla_\lambda)\}]$ is independent of the choices that were made. Thus, we have constructed a homomorphism $\tilde{H}^1(\hat{\mathcal{L}}, \hat{\nabla}) \to H^2(\mathcal{L}, \nabla)$ which can be checked to be bijective. □

Definition 8 Let $\tilde{H}^1_0(\hat{\mathcal{L}}, \hat{\nabla})$ be the subgroup of $\tilde{H}^1(\hat{\mathcal{L}}, \hat{\nabla})$ consisting of representatives such that $\hat{\omega}$ in eq. (4.2) satisfies the integrality condition with respect to manifolds contained in a fibre.

By lemma 5 this is equivalent to demanding that the integrality condition should be satisfied on sets $\pi^{-1}(U_\lambda)$.

Definition 9 Let $H_0(\hat{\mathcal{L}}, \hat{\nabla})$ be the subgroup of $\tilde{C}^1(\{\pi^{-1}(U_\lambda)\}, \hat{\mathcal{L}}, \hat{\nabla})$ with representatives such that $\hat{\nabla}^2 = \pi^*\nabla^2$ for some connection ∇ on a line bundle $\mathcal{L}$ over B.

Using theorem 1 and lemmas 6 and 7 the following statement is straightforward to prove.

Theorem 2

$$\begin{array}{ccccc}
H^1_{dR,0}(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}}) & \cong & \tilde{H}^1_0(\hat{\mathcal{L}}, \hat{\nabla}) & \xleftarrow{\text{monic}} & H_0(\hat{\mathcal{L}}, \hat{\nabla}) \\
\uparrow \text{monic} & & \uparrow \text{monic} & & \cong \\
H^2_{dR,0}(B, \underline{\mathbb{C}}) & \cong & H^2_0(\mathcal{L}, \nabla) & = & H^2_0(\mathcal{L}, \nabla)\,.
\end{array}$$

Let us now consider the case $k = 2$.

Lemma 8 $\tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla}) \cong \tilde{H}^2_0(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$

Proof. Let $[\{(\hat{\mathcal{L}}_\lambda, \hat{\nabla}_\lambda)\}] \in \tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla})$ be given. Let $\{\tilde{\omega}_\lambda\}$ be the curvature 2-form of $\hat{\mathcal{L}}_\lambda$. The cocycle relation for $\{(\hat{\mathcal{L}}_\lambda, \hat{\nabla}_\lambda)\}$ implies that $(\delta\tilde{\omega})_{\lambda\mu} = -\pi^* F_{\lambda\mu}$ for some $\{F_{\lambda\mu}\}$. However, since $(\delta F)_{\lambda\mu\nu} = 0$ we get $F_{\lambda\mu} = (\delta\omega)_{\lambda\mu}$. It follows that $(\delta(\tilde{\omega} + \pi^*\omega))_{\lambda\mu} = 0$ so the form $\hat{\omega}_\lambda := \tilde{\omega}_\lambda + \pi^*\omega_\lambda$ can be written as $\hat{\omega}|_{\pi^{-1}(U_\lambda)}$ for some globally defined form $\hat{\omega}$ on $\mathcal{A}$. Thus, $\{\tilde{\omega}_\lambda\}$ defines an element $[\{\tilde{\omega}_\lambda\}] \in \tilde{H}^2_0(\{\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$. It is easy to see that $[\{\tilde{\omega}_\lambda\}]$ is independent of the choice of representative $\{(\hat{\mathcal{L}}_\lambda, \hat{\nabla}_\lambda)\}$. The homomorphism $\tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla}) \to \tilde{H}^2_0(\mathcal{A} \stackrel{\pi}{\to} B, \underline{\mathbb{C}})$ that has been constructed is clearly injective and surjective. $\square$

As a corollary we obtain a monomorphism from $H^3_0(\mathcal{L}, \nabla)$ to $\tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla})$ as well. A direct proof of this could also have been made in a way similar to in the proof of lemma 7. The analogue of theorem 2 is:

Theorem 3

$$\begin{array}{ccc} \tilde{H}^2_0(\mathcal{A} \stackrel{\pi}{\to} B, \underline{\mathbb{C}}) & \cong & \tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla}) \\ \uparrow \text{monic} & & \uparrow \text{monic} \\ \check{H}^3_{dR,0}(B, \underline{\mathbb{C}})_{(1,2)} & \cong & H^3_0(\mathcal{L}, \nabla)\,. \end{array}$$

Let us summarize some of the essentials of this section. In lemma 2, the de Rham cohomology on B was "lifted" to an affine space $\mathcal{A}$. The representatives of $H^2_{dR,0}(B, \underline{\mathbb{C}})$ can be identified with 1-forms on $\mathcal{A}$. This gave a geometrical realization if considering the 1-forms as connection forms on a line bundle on $\mathcal{A}$. In the same way, the representatives of $H^3_{dR,0}(B, \underline{\mathbb{C}})$ can be identified with 2-forms on $\mathcal{A}$. The natural realization is curvature forms. Unfortunately though, the 2-forms are not necessary closed, while a curvature form of a line bundle is always closed due to the Bianchi identity. To solve this we use lemma 4 which gives us closed 2-forms. The price we have to pay is that we get a set of 2-forms, one on each $\pi^{-1}(U_\lambda)$. The geometrical realization of $H^3_{dR,0}(B, \underline{\mathbb{C}})$ is then in terms of curvatures on set of line bundles. Certainly, the same could have been done when $k = 1$, but this is not necessary since a connection form does not have to be closed. This can be stated in a different way: the second and the third columns from the right in theorem 1 can be used to obtain geometrical realizations for $k = 1$ respective $k = 2$. The fact that connection forms does not have to be closed makes it possible to use the right column for $k = 1$. It contains to some extent simpler groups.

When $k \geq 3$, things become more complicated. If one mimics the above approach, one is unfortunately forced to consider forms on $\mathcal{A}$ of at least degree 3. An alternative idea is to try to add a fourth column from the right in theorem 1, i.e. to find a lift of $\check{H}^{k+1}_{dR,0}(B, \underline{\mathbb{C}})_{(2,k-1)}$. Unfortunately, one would then run into problems which have their roots in the fact that $\{\pi^{-1}(U_\lambda)\}$ is not necessary a good covering. That $\{\pi^{-1}(U_\lambda)\}$ is not a good covering is the generic case, as we will see in the physical examples in the next section.

5. Anomalies and Schwinger terms

We will here show how the abstract constructions of the previous sections arise in a natural way from a physical model, at least for $k = 1$ and $k = 2$. For $k = 1$ we will consider the (non-abelian) chiral anomaly in space-time and for $k = 2$ the

Schwinger term. For this purpose we now let $\mathcal{A}$ be the affine space of (external) gauge connections and $B = \mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is the group of base-point preserving gauge transformations (so $\mathcal{A}/\mathcal{G}$ will be a smooth manifold). The corresponding gauge group G is assumed to be a compact semi-simple Lie group. Although we only consider gauge theories, everything works in parallel for gravitational anomalies and Schwinger terms, [17].

The generating "functional" for Weyl fermions can be written as

$$\exp(-W(A)) = \int d\psi d\bar{\psi} \exp\left(-\int_M \bar{\psi} \not{\partial}_A^+ \psi d^{2n}x\right),$$

where W is the effective action and $\not{\partial}_A^+$ is the positive chirality part of the anti-hermitean Dirac operator $\not{\partial}_A$. The space-time M is a compact, oriented and even-dimensional riemannian spin manifold without boundary. It has been argued that the generating functional can be identified (by using a reference section) with the canonical section $\det(\ker i\not{\partial}^+) \otimes (\det(\operatorname{coker} i\not{\partial}^+))^{-1}$ of the determinant line bundle $\widehat{\mathrm{DET}} i\not{\partial}$ over $\mathcal{A}$ (we assume that $\operatorname{ind} \not{\partial}^+ = 0$). The determinant line bundle has also a canonical metric, the Quillen metric. Associated with it is a natural connection $\nabla^{\widehat{\mathrm{DET}} i\not{\partial}}$ with curvature 2-form:

$$F^{\widehat{\mathrm{DET}} i\not{\partial}} = -2\pi i \left(\int_M \hat{a}(M)\, \mathrm{ch}(\mathcal{E})\right)_{[2]}, \tag{5.1}$$

see [18]. We use $(\cdot)_{[2]}$ to denote the part of the argument that is a 2-form. $\hat{a}$ is the a-roof function important for gravitation and the finite-dimensional vector bundle $\mathcal{E}$ is the tensor product of the spinor bundle and the gauge bundle. By pushing forward all structures from $\mathcal{A}$ to $\mathcal{A}/\mathcal{G}$, it is seen that $F^{\widehat{\mathrm{DET}} i\not{\partial}}$ can be written as $\pi^* F^{\mathrm{DET} i\not{\partial}}$. The curvature 2-form $F^{\mathrm{DET} i\not{\partial}}$ can be obtained by using the family index theorem on $\mathcal{A}/\mathcal{G}$, see [9]. We have thereby argued that $[(\widehat{\mathrm{DET}} i\not{\partial}, \nabla^{\widehat{\mathrm{DET}} i\not{\partial}})] \in H_0(\hat{\mathcal{L}}, \hat{\nabla})$.

An important object is the finite anomaly $\theta(A; g)$ which measures the lack of gauge invariance of the effective action:

$$W(A \cdot g) = W(A)\theta(A; g) \tag{5.2}$$

and the infinitesimal anomaly defined by:

$$\hat{\omega}(A; X) = \left.\frac{d}{dt}\right|_{t=0} \theta\left(A; e^{tX}\right),$$

where $g \in \mathcal{G}$ and $X \in \mathrm{Lie}(\mathcal{G})$. The correct interpretation in the above description is that the infinitesimal anomaly is the negative of the restriction of the connection on $\widehat{\mathrm{DET}} i\not{\partial}$ to gauge directions, i.e.

$$\nabla_X^{\widehat{\mathrm{DET}} i\not{\partial}} \hat{s}(A) = -\hat{s}(A)\hat{\omega}(A; X). \tag{5.3}$$

The minus sign originates from the fact that the generating functional is the exponential of minus the effective action (or from the choice of Dirac operator). From eq. (5.2) it follows that the anomaly has to satisfy the consistency condition. Furthermore, it is only well defined up to changes of $W(A)$ by local functionals in A, which implies that the anomaly can be regarded as an element in a cohomology class. In [19] it was proven that the cohomology class of the finite anomaly is isomorphic to $\check{H}^1(\mathcal{A}/\mathcal{G}, \underline{\mathbb{C}}^\times)$. The free part of this states that the cohomology class of the infinitesimal anomaly is isomorphic to $\check{H}^1_0(\mathcal{A}/\mathcal{G}, \underline{\mathbb{C}}^\times)$, or equivalently to $H_0(\hat{\mathcal{L}}, \hat{\nabla})$. Thus, the infinitesimal anomaly is described by $[(\widehat{\mathrm{DET}} i\partial\!\!\!/, \nabla^{\widehat{\mathrm{DET}} i\partial\!\!\!/})] \in H_0(\hat{\mathcal{L}}, \hat{\nabla})$ or equivalently, if we disregard torsion, by the equivalence class of line bundles on $\mathcal{A}/\mathcal{G}$ represented by $\mathrm{DET} i\partial\!\!\!/$. This is the well-known result by Atiyah and Singer [9].

An explicit expression for the anomaly can be obtained in the following way: first use the Poincaré lemma on eq. (5.1) to get a local expression (up to a coboundary) of minus the connection 1-form corresponding to $\nabla^{\widehat{\mathrm{DET}} i\partial\!\!\!/}$. The restriction of this 1-form to gauge directions then gives an expression for the anomaly. It is well known (and is easy to check) that this procedure is equivalent to the use of the descent equations.

We have thereby shown, in a physical example, that the cohomology groups in the previous sections are important when $k = 1$. Let us now show the correspondence for $k = 2$ and Schwinger terms. We will start with an intuitive discussion before going into the mathematical details. $\mathcal{A}$ and $\mathcal{G}$ will be defined with respect to a fixed time. Our starting point will be the fact that the Schwinger term can be identified with (minus) the curvature of the vacuum line bundle over $\mathcal{A}$. By vacuum is meant with respect to the filled up Dirac sea. It can thus be defined as the infinite wedge product of all eigenvalues of the hamiltonian H_A which have eigenvalues less than a real number λ, the vacuum level. The $\mathbb{C}$-span of the vacuum defines the vacuum line bundle Ω_λ. Clearly, it is only defined on $\hat{U}_\lambda = \{A \in \mathcal{A} | \lambda \notin \mathrm{spec}(H_A)\}$. Since the eigenvalues of H_A are independent of gauge transformations, it is possible to write $\hat{U}_\lambda = \pi^{-1}(U_\lambda)$ for some $U_\lambda \subset \mathcal{A}/\mathcal{G}$. The Schwinger term is clearly unaffected by a refinement of $\{U_\lambda\}$ and we can therefore assume that it is an open and good covering of $\mathcal{A}/\mathcal{G}$ (if such a covering exist, see comment at the end of the section). Since the Schwinger term arises from the curvature of the vacuum bundle, there must be some natural connection $\hat{\nabla}_\lambda$ associated with Ω_λ. Later, when defining the vacuum line bundle rigorously, we will see how this connection is constructed.

By applying $\hat{\delta}$ to $\{(\Omega_\lambda, \hat{\nabla}_\lambda)\}$ we obtain line bundles $\Omega_\lambda \otimes \Omega_\mu^{-1}$ and connections $\hat{\nabla}_\lambda - \hat{\nabla}_\mu$ over $\pi^{-1}(U_{\lambda\mu})$. From the intuitive definition of the vacuum bundle above it follows that $\Omega_\lambda \otimes \Omega_\mu^{-1}$ is the $\mathbb{C}$-span of the finite wedge product of all eigenvectors with eigenvalues in the interval $[\lambda, \mu]$. We will denote this line bundle by $\widehat{\mathrm{DET}}_{\lambda\mu}$. From the gauge independence of the eigenvalues of H_A it follows that $\widehat{\mathrm{DET}}_{\lambda\mu}$ can be naturally pushed forward to $U_{\lambda\mu} \subset \mathcal{A}/\mathcal{G}$. Thus, if the connection $\hat{\nabla}_\lambda$ should be natural, it must be such that $((\hat{\delta}\Omega)_{\lambda\mu}, (\hat{\delta}\hat{\nabla})_{\lambda\mu})$ is equivalent with some $(\pi^* \mathrm{DET}_{\lambda\mu}, \pi^* \nabla_{\lambda\mu})$.

The reason why $(\Omega_\lambda, \hat{\nabla}_\lambda)$ can not be pushed forward is that the vacuum bundle is spanned by the infinite (and not finite) wedge product of eigenvectors. We have thereby argued that $[\{(\Omega_\lambda, \hat{\nabla}_\lambda)\}] \in \tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla})$. In fact, as we will see later, it defines an element in the image of $H^3_0(\mathcal{L}, \nabla)$ under the monomorphism in theorem 3. It is thus possible to use homomorphisms from previous sections. For instance, it is the image of the right monomorphism in theorem 3 that defines the line bundles $\mathrm{DET}_{\lambda\mu}$ with connections $\nabla_{\lambda\mu}$.

The fact that the Schwinger term can be identified with (minus) the restriction of the curvature $(\hat{\nabla}_\lambda)^2$ to gauge directions defines a homomorphism from $\tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla})$ to the relevant cohomology (the one in the descent equations) of the Schwinger term. Non-triviality of the Schwinger term can thus be interpreted as coming from a non-zero element in $H^3_0(\mathcal{L}, \nabla)$, i.e. from a set $\{(\mathrm{DET}_{\lambda\mu}, \nabla_{\lambda\mu})\}$ such that the line bundles $\{\mathrm{DET}_{\lambda\mu}\}$ do not "glue together" due to the connections $\{\nabla_{\lambda\mu}\}$. This is the analog of Atiyah and Singer's geometrical interpretation of anomalies which in our language states that local trivial line bundles do not glue together to a global trivial bundle due to certain connections. We have thus generalized the torsion-free part of Atiyah and Singer's geometrical interpretation of anomalies to the case of Schwinger terms.

Let us now develop the above ideas in a strict manner. A mathematical definition of the vacuum line bundle will be needed. It must also have a natural connection so that an element in $\tilde{H}^2(\hat{\mathcal{L}}, \hat{\nabla})$ is obtained.

The Hilbert space H of fermion wave functions is the space of square integrable sections of a vector bundle $\mathcal{E}$ with base M. By abuse of notation we will use M and $\mathcal{E}$ to denote the physical space (no time) and the corresponding vector bundle. The hamiltonian H_A defines a decomposition $H = H_+(A,\lambda) \oplus H_-(A,\lambda)$ over $\pi^{-1}(U_\lambda)$, where $H_-(A,\lambda)$ is the Hilbert space spanned by all eigenvalues of H_A with eigenvalues less than λ. Above, we used the intuitive definition that the vacuum was the wedge product of a complete set of basis vectors in $H_-(A,\lambda)$. This definition is bad from a mathematical perspective since it is the wedge product of an infinite number of vectors. One way out of this is to "subtract" a fixed wedge product corresponding to a reference vacuum so that only a finite numbers of wedge products would be left. We will use $H_-(A_0,\lambda_0)$, for some fixed $A_0 \in \pi^{-1}(U_{\lambda_0})$ and $\lambda_0 \in \mathbb{R}$ to define this reference vacuum. To define the vacuum corresponding to $H_-(A,\lambda)$ we must have a way to compare it with $H_-(A_0,\lambda_0)$. We thus want to fix a connection in the trivial Hilbert space bundle over $\mathcal{A}$ with fibre $H(A) = H$ at $A \in \mathcal{A}$. The canonical identification of $H(A)$ and $H(A_0)$ given by the fact that both spaces are equal to H has been used in for instance [20] to compute the Schwinger term. This gives the usual renormalization problem with Schatten classes (Hilbert-Schmidt operators in 1 space dimension). In [8] it was shown how this can be avoided by using an alternative identification which is natural from a physical perspective. We will now review this procedure.

Let $A(t) = f(t)A + (1 - f(t))A_0$, $t \in I = [0,1]$, where f is a smooth function which is 0 when $t \in [0,\epsilon]$ and 1 when $t \in [1-\epsilon, 1]$, for some $\epsilon \in (0,1/2)$. The Dirac equation $i\partial_t\psi = H_{A(t)}\psi$, can then be used to identify vectors in $H(A)$ with vectors in $H(A_0)$. $H_-(A,\lambda)$ is mapped to a subspace $H_-(A,\lambda)_{A_0} \subset H(A_0)$. Consider the spaces

$$\begin{aligned} H_{\lambda,+-}(A) &= H_+(A_0,\lambda_0) \cap H_-(A,\lambda)_{A_0}\,, \\ H_{\lambda,-+}(A) &= H_-(A_0,\lambda_0) \cap H_+(A,\lambda)_{A_0}\,. \end{aligned}$$

We will soon show that they are finite dimensional. This leads to vector bundles $H_{\lambda,+-}$ and $H_{\lambda,-+}$ over a subset of $\mathcal{A}$. By wedging together all vectors in $H_{\lambda,+-}(A)$ and $H_{\lambda,-+}(A)$ we obtain line bundles $\widehat{\mathrm{DET}}_{\lambda,+-}$ and $\widehat{\mathrm{DET}}_{\lambda,-+}$, respectively. It is then natural to define Ω_λ as $\widehat{\mathrm{DET}}_{\lambda,+-} \otimes (\widehat{\mathrm{DET}}_{\lambda,-+})^{-1}$. This is to be thought of as the "subtraction" of the infinite wedge product of the basis vectors in $H_-(A,\lambda)$ by the corresponding product with respect to the reference space $H_-(A_0,\lambda_0)$. Ω_λ is however not well defined on all of $\pi^{-1}(U_\lambda)$ since the dimensions of $H_{\lambda,+-}(A)$ and $H_{\lambda,+-}(A)$ are not necessary constant. To avoid this problem we will once again redefine the vacuum bundle.

We will now reinterpret the identification of vectors in $H(A_0)$ and $H(A)$ by using vectors ψ in the Hilbert space $H \times I$ of square integrable sections of the vector bundle $\mathcal{E} \times I$ over $M \times I$. The identification then comes from the ψ's which satisfies $i\partial\!\!\!/_A^+\psi = 0$ where $i\partial\!\!\!/_A^+ = i\partial_t - H_{A(t)}$. We will choose the following boundary conditions for the partial differential equation: $\psi|_{t=0} \in H_+(A_0,\lambda_0)$ and $\psi|_{t=1} \in H_-(A,\lambda)$. By restriction to $t=0$ it is clear that $\ker i\partial\!\!\!/^+$ can be canonically identified with $H_{\lambda,+-}$. Similar, $\operatorname{coker} i\partial\!\!\!/^+$ can be identified with $H_{\lambda,-+}$. Our setting is now identical to the one used for determinant line bundles for manifolds with boundaries, see [21]. We can therefore use the powerful results that have been achieved there. For instance, the fact that $\ker i\partial\!\!\!/^+$ and $\operatorname{coker} i\partial\!\!\!/^+$ are finite dimensional. Also, we can resolve the "jump" in the dimensions by using an idea due to Quillen. The result is a well-defined line bundle $\widehat{\mathrm{DET}}_\lambda$ on $\pi^{-1}(U_\lambda)$. $\widehat{\mathrm{DET}}_\lambda$ can be canonically identified with $\det(\ker i\partial\!\!\!/^+) \otimes (\det(\operatorname{coker} i\partial\!\!\!/^+))^{-1}$ when there are no "jumps". It also follows that $\widehat{\mathrm{DET}}_\lambda$ has a natural metric, the Quillen metric. According to the first reference in [18], there is a natural connection $\nabla^{\widehat{\mathrm{DET}}_\lambda}$, compatible with the Quillen metric. This motivates us to define $(\Omega_\lambda, \hat{\nabla}_\lambda)$ as $(\widehat{\mathrm{DET}}_\lambda, \nabla^{\widehat{\mathrm{DET}}_\lambda})$.

We have now succeeded in finding a rigorous definition of the vacuum line bundle. In addition, we have seen that it is equipped with a natural connection. What is left to show is that $\{(\Omega_\lambda, \hat{\nabla}_\lambda)\}$ defines an element in $\tilde{H}^2_0(\hat{\mathcal{L}}, \hat{\nabla})$. We will also show how an explicit expression for the Schwinger term can be obtained.

The expression

$$F^{\widehat{\mathrm{DET}}_\lambda} = -2\pi i \left(\int_{M\times I} \hat{a}(M\times I)\,\mathrm{ch}(\mathcal{E}\times I)|_{\pi^{-1}(U_\lambda)} - \frac{1}{2}\hat{\eta}_\lambda \right)_{[2]} \tag{5.4}$$

for the curvature 2-form corresponding to $\nabla^{\widehat{\mathrm{DET}}_\lambda}$ was calculated in [21]. The only thing that we will need about $\hat{\eta}_\lambda$ is that it is defined from the boundary spectral data. It is therefore independent of gauge transformations and can thus be written as $\pi^*\omega_\lambda$ for some 2-form ω_λ on U_λ. Since the first term on the right hand side of eq. (5.4) is the restriction of a form that is independent of the boundary conditions, i.e. independent of λ, we get that $F^{\widehat{\mathrm{DET}}_\lambda}$ represents an element in $\tilde{H}^2(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$. Notice that

$$\begin{aligned}
\hat{d}(-2\pi i)\int_{M\times I} \hat{a}(M\times I)\,\mathrm{ch}(\mathcal{E}\times I) &= -2\pi i\int_{M\times I}(\hat{d}+d_{M\times I}-d_{M\times I})\times \\
&\qquad\qquad \times\, \hat{a}(M\times I)\,\mathrm{ch}(\mathcal{E}\times I) \\
&= 2\pi i\int_{M\times\partial I} \hat{a}(M\times \partial I)\,\mathrm{ch}(\mathcal{E}\times \partial I)\,.
\end{aligned}$$

Just as the form in (5.1), the 3-form part of this can be pushed forward to $\mathcal{A}/\mathcal{G}$. The integral over any closed 3-manifold then gives an element in $2\pi i\mathbb{Z}$ by the index theorem. This proves that $F^{\widehat{\mathrm{DET}}_\lambda}$ in fact defines an element in $\tilde{H}^2_0(\mathcal{A} \xrightarrow{\pi} B, \underline{\mathbb{C}})$ and that $[\{(\Omega_\lambda, \hat{\nabla}_\lambda)\}]$ is in the image of $H^3_0(\mathcal{L}, \nabla)$ under the monomorphism in theorem 3. We can therefore use the results we arrived at in the earlier intuitive discussion. Thus, $(\Omega_\lambda, \hat{\nabla}_\lambda)$ defines $(\mathrm{DET}_{\lambda\mu}, \nabla_{\lambda\mu})$ and the Schwinger terms appear since the trivial line bundles $\mathrm{DET}_{\lambda\mu}$ do not "glue together" due to the connections $\nabla_{\lambda\mu}$.

We will now indicate how the Schwinger term can be computed. We thus restrict eq. (5.4) to gauge directions. The form $\hat{\eta}_\lambda$ will then give zero contribution. We need to compute $\mathrm{ch}(\mathcal{E}\times I)_{[2m]} = (i/2\pi)^m \,\mathrm{tr}\,\mathcal{F}^m/m!$, where $\mathcal{F}$ is the curvature of the bundle $\mathcal{E}\times I\times\mathcal{A} \to M\times I\times\mathcal{A}$. The connection on this bundle is $A(t)$ when restricted to the the first two factors. We will only need the connection in gauge directions for the last factor. Following [9] we demand that the connection should be invariant under gauge transformations of the bundle. This gives the connection $f(t)A + (1-f(t))A_0 + f(t)v = A(t) + f(t)v$, where v is the ghost. It is important to remember that the gauge transformations are at a fixed time, i.e. with respect to $\mathcal{A}$, and not $\mathcal{A}\times I$. Inserting this into eq. (5.4) and performing the integration over I gives then the Schwinger term. Certainly, the computation is independent of the choice of f. When $A_0 = 0$ it gives the usual expression for the Schwinger term, which can be obtained for example by the descent equations. A non-zero A_0 gives the expression for the Schwinger term with a background connection.

There is an interesting relation between the terms in eq. (5.4). The first term on the right-hand side appears also for the case when manifolds without boundary are considered. It is the second term which "takes care" of the boundary conditions. From this point of view the two terms seem unrelated. Observe now that eq. (5.4) can be identified with eq. (4.1). Lemma 4 then states that if we know the cohomology class of one of the three terms in eq. (5.4), then we also know it for the other two. For example, knowing the relevant cohomology class of $-2\pi i(\int_{M\times I}\hat{a}(M\times I)\,\mathrm{ch}(\mathcal{E}\times I))_{[2]}$, we also know it for the set $\{d\omega_\lambda\}$, where $\hat{\eta}_\lambda = \pi^*\omega_\lambda$. This shows for instance that, at the level of cohomology, we never need to know the explicit expression for $\hat{\eta}_\lambda$ in this case.

In this section we have put $B = \mathcal{A}/\mathcal{G}$ and used statements from the previous sections. This means that we implicitly assume things about $\mathcal{A}/\mathcal{G}$. For instance that it admits a good covering and a partition of unity. The existence of a partition of unity was shown in [8] by the use of a theorem by Milnor. However, that $\mathcal{A}/\mathcal{G}$ admits a good covering might not be true. That is not so crucial for this section since many of the main ideas goes through even if the covering is not good. It is straightforward to check that we then obtain the Čech cohomology $\check{H}^k(\{U_\lambda\}, \underline{\mathbb{C}}^\times)$ that depends on the covering and the corresponding cohomology $H^k(\{U_\lambda\}, \mathcal{L}, \sigma)$, discussed at the end of section 2.

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A. Proof of proposition 2

That $[F] \in H^{k+1}_{dR,0}(B, \underline{\mathbb{C}})$ is equivalent with that F up to a (uninteresting) coboundary can be written as

$$F_\lambda = \sum_{\lambda_0,\ldots,\lambda_{k-1}} d\rho_{\lambda_0} \wedge d\rho_{\lambda_1} \wedge \cdots \wedge d\rho_{\lambda_{k-1}} \wedge d\log g_{\lambda\lambda_0\lambda_1\ldots\lambda_{k-1}}$$

over U_λ, where $[\{g_{\lambda_0\ldots\lambda_k}\}] \in \check{H}^k(B, \underline{\mathbb{C}}^\times)$. We will now show that F satisfies the integrality condition, i.e. that its integral over any closed $(k+1)$-dimensional manifold S equals $2\pi i$ times an integer. For this purpose we assume, without loss of generality, that there exist a finite subset $\{U_\lambda\}_{\lambda\in\Lambda_0}$ such that $V_\lambda := U_\lambda \cap S$ are open, disjoint sets with $\dim(\partial V_{\lambda_{i_0}} \cap \cdots \cap \partial V_{\lambda_{i_l}}) = k-l+1$ and $S = \cup_{\Lambda\in\Lambda_0} \overline{V_\lambda}$, where the bar denotes the closure. Then,

$$\begin{aligned}
\int_S F &= \sum_{\mu_0,\lambda_0,\ldots,\lambda_{k-1}} \int_{V_{\mu_0}} d\rho_{\lambda_0} \wedge d\rho_{\lambda_1} \wedge \cdots \wedge d\rho_{\lambda_{k-1}} \wedge d\log g_{\mu_0\lambda_0\lambda_1,\ldots,\lambda_{k-1}} \\
&= \sum_{\mu_0,\lambda_0,\ldots,\lambda_{k-1}} \int_{\partial V_{\mu_0}} \rho_{\lambda_0} \wedge d\rho_{\lambda_1} \wedge \ldots \wedge d\rho_{\lambda_{k-1}} \wedge d\log g_{\mu_0\lambda_0\lambda_1,\ldots,\lambda_{k-1}} \\
&= \frac{1}{2!} \sum_{\mu_0,\mu_1,\lambda_0,\ldots,\lambda_{k-1}} \int_{\partial V_{\mu_0}\cap\partial V_{\mu_1}} \rho_{\lambda_0} \wedge d\rho_{\lambda_1} \wedge \cdots \wedge d\rho_{\lambda_{k-1}} \wedge \\
&\qquad\qquad \wedge d(\log g_{\mu_0\lambda_0\lambda_1,\ldots,\lambda_{k-1}} - \log g_{\mu_1\lambda_0\lambda_1,\ldots,\lambda_{k-1}}) \\
&= \frac{1}{2!} \sum_{\mu_0,\mu_1,\lambda_1,\ldots,\lambda_{k-1}} \int_{\partial V_{\mu_0}\cap\partial V_{\mu_1}} d\rho_{\lambda_1} \wedge \cdots \wedge d\rho_{\lambda_{k-1}} \wedge d\log g_{\mu_0\mu_1\lambda_1,\ldots,\lambda_{k-1}} \,,
\end{aligned}$$

where the cocycle condition was used in the last step. The manifold $\partial V_{\mu_0} \cap \partial V_{\mu_1}$ is assumed to have orientation given by ∂V_{μ_0}. This explains the relative sign in the fourth expression above. Repeating the above process until we get rid of all the partitions of unity gives:

$$\begin{aligned}\int_S F &= \frac{1}{(k+1)!} \sum_{\mu_0,\mu_1,\ldots,\mu_{k+1}} \log g_{\mu_0\mu_1,\ldots,\mu_k}|_{\partial V_{\mu_0}\cap\partial V_{\mu_1}\cap\cdots\cap\partial V_{\mu_{k+1}}} \\ &= (-1)^{k+1} \sum_{\mu_0<\mu_1<\cdots<\mu_{k+1}} \sum_{i=0}^{k+1} (-1)^i \log g_{\mu_0,\ldots,\hat{\mu}_i,\ldots,\mu_{k+1}}\big|_{\partial V_{\mu_0}\cap\partial V_{\mu_1}\cap\cdots\cap\partial V_{\mu_{k+1}}} \in 2\pi i\mathbb{Z}\,.\end{aligned}$$

The proof of the converse statement can be performed on each path connected component of B. We can thus without loss of generality assume that $\pi_0(B) = 0$. Let f_l^m be a choice of representative for each element in $\pi_l(B)$, $l = 0, 1, \ldots$ Then f_l^m is a continuous map from the l-dimensional sphere to B. Certainly, we can assume that the sets U_λ are star-shaped. Now choose reference points $b_0 \in B$, $b_\lambda \in U_\lambda, \forall\lambda$ and paths γ_λ from b_0 to b_λ. Let $b \in U_{\lambda_0\ldots\lambda_k}$ and choose for each $i = 0, 1, \ldots, k$ a path $\triangle^b_{\lambda_i}$ from b_{λ_i} to b which depends smoothly on b. Then, for $j \neq i$, $[\gamma_{\lambda_i} \cup \triangle^b_{\lambda_i} \cup (\triangle^b_{\lambda_j})^{-1} \cup (\gamma_{\lambda_j})^{-1}] = [f_1^m] \in \pi_1(B)$ for some $f_1^m = f_1^{m(i,j)}$. For all $i \neq j$ then choose 2-manifolds $\triangle^b_{\lambda_i\lambda_j}$ which have $\gamma_{\lambda_i} \cup \triangle^b_{\lambda_i} \cup (\triangle^b_{\lambda_j})^{-1} \cup (\gamma_{\lambda_j})^{-1} \cup (f_1^m)^{-1}$ as its boundary and depends smoothly on b. For $l \neq i, j$, we then have $[\triangle^b_{\lambda_j\lambda_l} \cup (\triangle^b_{\lambda_i\lambda_l})^{-1} \cup \triangle^b_{\lambda_i\lambda_j}] = [f_2^m] \in \pi_2(B)$, and can thus choose 3-manifolds $\triangle^b_{\lambda_i\lambda_j\lambda_l}$ with $\triangle^b_{\lambda_j\lambda_l} \cup (\triangle^b_{\lambda_i\lambda_l})^{-1} \cup \triangle^b_{\lambda_i\lambda_j} \cup (f_2^m)^{-1}$ as boundary. $(\triangle^b_{\lambda_i\lambda_l})^{-1}$ is the same manifold as $\triangle^b_{\lambda_i\lambda_l}$, but with the opposite orientation. Continue this procedure until $(k+1)$-manifolds $\triangle^b_{\lambda_0\ldots\lambda_k}$ has been chosen. Then $\partial\triangle^b_{\lambda_0\ldots\lambda_k} = (\cup_{i=0}^k {\triangle^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k}}^{(-1)^i}) \cup (f_k^m)^{-1}$. From F we can now construct the corresponding holonomy-free $(k-1)$-gerbe by

$$g_{\lambda_0\ldots\lambda_k}(b) = \exp\left(\int_{\triangle^b_{\lambda_0\ldots\lambda_k}} F\right).$$

The cocycle property of $\{g_{\lambda_0\ldots\lambda_k}\}$ is a consequence of the integrality condition. The proof follows if we can prove that $[F]$ is the image of $[\{g_{\lambda_0\ldots\lambda_k}\}]$ under the map from $\check{H}^k(B, \underline{\mathbb{C}}^\times)$ to $H^{k+1}_{dR}(B, \underline{\mathbb{C}})$. Writing F as $F_\lambda = \sum_{\lambda_0,\ldots,\lambda_{k-1}} d\rho_{\lambda_0} \wedge d\rho_{\lambda_1} \wedge \cdots \wedge d\rho_{\lambda_{k-1}} \wedge \omega_{\lambda\lambda_0\lambda_1\ldots\lambda_{k-1}}$ we see that we must show that $\{\omega_{\lambda_0\ldots\lambda_k}\}$ and $\{d\log g_{\lambda_0\ldots\lambda_k}\}$ represents the same element in the Čech-de Rham cohomology. To prove this we will make use of the fact that the cohomology class of $\{g_{\lambda_0\ldots\lambda_k}\}$ is independent of the choices of b_0, b_λ, γ_λ and $\triangle^b_{\lambda_{i_0}\ldots\lambda_{i_l}}$. For example, since F fulfills the integrality condition it is possible to replace $\triangle^b_{\lambda_0\ldots\lambda_k}$ with any other manifold which has the same boundary. If for $i = 0, \ldots, k$ we change $\triangle^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k}$ to $\triangle'^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k}$ then

$$\int_{\triangle'^b_{\lambda_0\ldots\lambda_k}} F = \int_{\triangle^b_{\lambda_0\ldots\lambda_k}} F + \sum_{i=0}^k (-1)^i \int_{W^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k}} F\,.$$

Here $\partial W^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k} = (\triangle^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k})^{-1} \cup \triangle'^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k} \cup f^m_{k-1}$, where f^m_{k-1} represents the same element in $\pi_{k-1}(B)$ as $(\triangle^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k})^{-1} \cup \triangle'^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k}$. This only changes $\{g_{\lambda_0\ldots\lambda_k}\}$ by a coboundary. A change in b_0, b_λ, γ_λ and $\triangle^b_{\lambda_{i_0}\ldots\lambda_{i_l}}$, $l \leq k-2$, does also only give a coboundary. The reason is that we can choose the $\triangle^b_{\lambda_0\ldots\hat{\lambda}_i\ldots\lambda_k}$'s such that the measure of $\triangle^b_{\lambda_0\ldots\lambda_k}$ is unaffected by such variations (as long as the variations do not affect the point $\{b\}$). The manifolds b_0, b_λ, γ_λ and $\triangle^b_{\lambda_{i_0}\ldots\lambda_{i_l}}$, $l \leq k-2$, are thus of too low dimension to cause any changes in $\triangle^b_{\lambda_0\ldots\lambda_k}$'s measure. With similar methods it can also be proven that the cohomology class of $\{g_{\lambda_0\ldots\lambda_k}\}$ is independent of the choice of representatives f^m_l.

The above discussion implies that we may assume that the b-dependence of $g_{\lambda_0\ldots\lambda_k}$ is concentrated to a small contractible neighbourhood $N \subset U_{\lambda_0\ldots\lambda_k} \cap \triangle^b_{\lambda_0\ldots\lambda_k}$ of $\{b\}$. Let $V_{\lambda_0}, \ldots, V_{\lambda_{k+1}}$ be open, disjoint sets with $\dim(\partial V_{\lambda_{i_0}} \cap \ldots \cap \partial V_{\lambda_{i_l}}) = k - l + 1$, $N \subset \cup_{i=0}^k \overline{V_{\lambda_i}} \subset U_{\lambda_0\ldots\lambda_k} \cap \triangle^b_{\lambda_0\ldots\lambda_k}$ and $\{b\} \in \partial(\partial V_{\lambda_0} \cap \ldots \cap \partial V_{\lambda_k})$. Then,

$$\begin{aligned} d\log g_{\lambda_0\ldots\lambda_k}(b) &= \sum_{i=0}^{k+1} d\int_{V_{\lambda_i}} F_{\lambda_i} \\ &= \sum_{i=0}^{k+1} \sum_{\mu_0,\ldots,\mu_{k-1}} d\int_{\partial V_{\lambda_i}} \rho_{\mu_0} \wedge d\rho_{\mu_1} \wedge \cdots \wedge d\rho_{\mu_{k-1}} \wedge \omega_{\lambda_i\mu_0\mu_1,\ldots,\mu_{k-1}} = \cdots \\ &= d\int_{\partial V_{\lambda_0}\cap\cdots\cap\partial V_{\lambda_k}} \omega_{\lambda_0\ldots\lambda_k}\,, \end{aligned}$$

where in the last step we used the computational techniques from the first part of the proof. Notice however that $\cup_{i=0}^{k+1}\overline{V_{\lambda_i}}$ is not a closed manifold here. Equalities such as $\partial V_{\lambda_i} = \sum_{j\neq i} \partial V_{\lambda_i} \cap \partial V_{\lambda_j}$ are therefore no longer true. However, the above equation follows anyway since the extra terms that appear do not depend on $\{b\}$. Since $\omega_{\lambda_0\ldots\lambda_k}$ is exact on N, the right hand side equals $\omega_{\lambda_0\ldots\lambda_k}(b)$, which was to be proven. □

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